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Skills Diversity in Unity

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At any point in time, skills gaps, mismatches, and shortages arise because of an imperfect correspondence between the singular sets of skills required by different open vacancies and the unique combinations of capabilities embodied in every job seeker - *skills diversity in unity*. This paper first constructs an abstract framework for defining and thinking about these phenomena in a unified, formal and objective way. The main building block is a discrete skills space in which the locations of vacancies and workers are determined by the vectors of skills characterizing them. We define skills gaps and mismatches as two different distance measures between them, and derive a condition for each vacancy that determines whether or not it experiences a skills shortage. We then develop a job matching model with imperfect information, in which skills mismatches influence the job application decisions of the workers, while skills gaps and shortages shape the competition for workers on the resulting bipartite job applications network. The tools proposed in this paper could in future work be employed as the main ingredients of an agent-based model used to investigate how skills gaps, mismatches and shortages affect equilibrium outcomes in the context of *skills diversity in unity* and imperfect information.

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SKILLS DIVERSITY IN UNITY

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1. Introduction

> *"At bottom every man knows well enough that he is a unique being, only once on this earth; and by no extraordinary chance will such a marvelously picturesque piece of diversity in unity as he is, ever be put together a second time."*
>
> Friedrich Nietzsche [28]

There has been a lot of debate around the notions of skills gaps, mismatches, and shortages. Some academics completely deny these issues, for instance disparaging skills gaps as a "zombie idea" (Krugman [22]) or "employer whining" (Cappelli [5]).

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At the same time, however, numerous surveys conducted by governmental bodies (e.g. the European Commission [8], the UK Commission for Employment and Skills (UKCES) [18]), lobbying organisations (e.g. the Confederation of British Industry (CBI) [6]), and consulting companies (e.g. KPMG [6], ManpowerGroup [25], Hays and Oxford Economics [16]), have been reporting skills gaps and shortages as main obstacles to business operations and a "handbrake on global growth" (Cox [16]) for years. Both public and private sectors spend large amounts of money on investigating and trying to reduce them. For example, in 2013, J.P. Morgan Chase launched a $250 million initiative *"New Skills at Work"*, with their CEO Jamie Dimon quoting the nearly 11 million unemployed Americans and the concurrent 4 million unfilled jobs as evidence of there being a "gulf between the skills job seekers currently have and the skills employers need to fill their open positions"[10]. Such concerns also play an important role in shaping migration policies - e.g. the UKCES reviews were commissioned by the UK Migration Advisory Committee (MAC). Hence, the evidence is not just a "telephone survey [with] executives" (Krugman [22]).

On the workers' side, many labour economists and special institutions (e.g. the European Centre for the Development of Vocational Training (Cedefop) [13]) have extensively studied and documented skills mismatches, the related phenomena of under and over-education (McGuinness [26], Sattinger [35]), and their dire consequences for wages, job satisfaction, and career prospects (e.g. Allen & Van der Velden [2]). In the UK, for instance, according to the ONS [12], 47% of recent graduates were in non-graduate employment in 2013, and the figure was already high even before the recession, e.g., at 43% in 2007. This constitutes a substantial waste of resources and leads to the following puzzle: why aren't job seekers acquiring the skills needed by employers, thereby eliminating skills gaps, mismatches, and shortages?

The proponents of the idea that skills gaps, mismatches, and shortages are just a "myth" add to the perplexity by saying that if these phenomena did exist in reality, we would observe tight labour market conditions (high wages, low unemployment rates) for those workers who have the scarce skills, but we do not. In a recent article, Shierholz [36] shows evidence for the USA that "unemployed workers dramatically outnumber job openings in all sectors" and "in no occupation is there any hint of wages being bid up in a way that would indicate tight labor markets or labor shortages". According to her, it is therefore "aggregate demand", and not structural skills issues, that is behind the weak job recoveries and high non-graduate employment levels of recent graduates.

Although this view is plausible, it seems to ignore the fact that despite recession-related rises, the numbers and concerns around skills gaps, shortages, and mismatches have been high throughout the business cycle, and this persistence over time might actually be the reason why the "zombie idea [...] refuses to die" (Krugman, [22]).

Perhaps all the debate about the existence and importance of skills gaps, shortages, and mismatches is the result of the ambiguity in their definitions, and the little attention paid to the notion of *skills diversity in unity*. The easiest way to understand the latter idea, is to talk to an actual recruiter who might tell you a story about overqualified applicants not being hired because they exhibited a lack of communication skills during interviews. In general, the "lack" is more substantial, but the key intuition is the same. At any point in time, skills gaps, mismatches, and shortages arise because of an imperfect correspondence between the singular sets of skills required by different open vacancies and the unique combinations of capabilities embodied in every job seeker.

Our paper aims to contribute to this debate by proposing tools, which, in future work, could be employed as the main ingredients of an agent-based model used to investigate how skills gaps, mismatches and shortages affect equilibrium outcomes in the context of skills diversity in unity and imperfect information.

In Section 3, we develop an abstract framework for defining skills gaps, mismatches and shortages geometrically and thinking about these phenomena in a unified, formal way. The main building block is a discrete skills space in which the locations of vacancies and workers are determined by the vectors of skills characterizing them. We define skills gaps and mismatches as two different distance measures between them. Conceptualising skills shortages - which occur "when there are not enough people available with the skills needed to do the jobs which need to be done" (British Government's Training Agency [1]) - is more complex. The "not enough" notion implies that skills shortages are not pairwise independent like skills gaps and mismatches. Hence, their existence for different vacancies, and the policies aimed at eliminating them cannot be considered in isolation. This highlights the importance and advantage of using a measurable skills space which directly accounts for interdependencies in a given economy, and provides a clear condition for each vacancy that determines whether or not it experiences a skills shortage. We also show how to determine minimum levels of skills mismatches and skills gaps achievable in an economy if the goal is to simultaneously reduce the number of unmatched agents (unemployed workers and unfilled vacancies).

The second part (Section 4) develops a two-sided job matching model with imperfect information, in which skills mismatches influence the job application decisions of the workers, while skills gaps and shortages shape the competition for workers on the resulting bipartite job applications network. A preliminary R code file for simulating the spatial structure and the competitive wage adjustment mechanism is available on request.

2. Related Literature

This paper can be related to different areas of theoretical and empirical literature in economics and networks, and management (operations research). In this section, we outline the main differences and similarities, and motivate our modelling approach.

The first papers in economics that pay attention to the notion of skills are Roy [33] and Tinbergen [39], both published in 1951. Tinbergen's discussion of skills heterogeneity is close to the one we present below. He recognizes that "many types of employment [...] require certain abilities in varying degrees", so that "in reality, [...] multi-dimensional descriptions of the nature of occupations [...] have to be considered". However, despite these statements, Tinbergen then assumes that "the nature of the labour required is a one-dimensional quantity", summarized by "one number s", and interpreted as "physical effort". Since Tinbergen intends to investigate the distribution of labour incomes, this simplification seems appropriate and useful in order to get analytical results. However, for the analysis of skills mismatches, gaps, and shortages, it is too restrictive because it leads to ignoring a scenario where people have multiple skills and hence different abilities in performing different jobs. For instance, worker A may be more productive in job X than worker B, while B would be more productive than A in another job Y.

Roy [33] was the first economist to clearly understand this, and also to recognise that this implies self-selection: A would choose to work in job X, while B - in job Y. The main difference between our conception of skills heterogeneity and Roy's, is that in Roy's multiple-index model workers have several types of skills but can only use one skill at a time depending on which occupation they choose. By contrast, we assume that both workers and jobs are characterized by multi-dimensional vectors of skills, and production is decreasing in the skills gap between the skills required by the vacancy and those possessed by the worker.

The main contribution of these two early papers is that they generated a substantial literature in which skills diversity plays a key role.

Assignment models that started with Sattinger [34][1] assume infinite numbers of worker and job types. However, as in Tinbergen [39], heterogeneity in these models is typically defined along one dimension only: from low to high ability for workers, and from easy to complex for jobs/tasks. Another difference with our approach is that these models usually require perfect information about all employers' wage offers and all workers' abilities. As discussed below, we assume the worker does not observe the

[1]Teulings & Vieira [38] show how assignment models can be estimated, while Shimer [37] proposes a version with coordination frictions.

job requirements and the wage offers of all employers; he just applies to a vacancy with some probability that is increasing in the ex-ante (before any network-induced competition for him) utility that he would get if employed in this position. Similarly, employers are unaware of the distribution of workers, but they can perfectly observe the skills of those who do apply for their vacancy (by, for instance, inviting the candidates to interviews).

Despite these differences, assignment models are close to our approach to the extent that skills diversity in them is also the main driving force behind the matching of workers and jobs. In particular, the allocation of workers to jobs in assignment models is governed by heterogeneity in either productivities (more able workers have a comparative advantage in more complex jobs, cf. Sattinger [34]), or preferences (workers have different tastes for performing diverse tasks, cf. Tinbergen [40]). In our model, skills mismatches influence job application decisions, while skills gaps and shortages shape the competition for workers on the resulting bipartite job applications network.

The skills space we use to model skills diversity and define skills gaps, mismatches, and shortages is similar in some respects to the "characteristics" space that forms the basis of hedonic models. Although the first instances of such models were developed to study differentiated products (Lancaster [24], Rosen [30]), the same approach has later been applied to labour markets (e.g. Heckman and Scheinkman [17]). The main idea is that products (workers) are "collections of characteristics" (Lancaster [24]) that yield utility (productive efficiency). By assuming n sectors with different production functions, Heckman and Scheinkman [17] also introduce heterogeneity on the labour demand side. However, by contrast with our approach, they do not directly map their jobs onto the same characteristics space as the one used to conceptualize their workers and do not explicitly model how the different measures of divergences between the skills vectors supplied and those demanded affect utility and production.

Perhaps the most important difference between the models discussed so far and the approach we take in this paper lies in the pricing of skills. In both hedonic and assignment models, the prices of different types of skills (characteristics) are determined by equilibrium between their supplies and demands. Although Heckman and Scheinkman [17] show that "whenever population skill endowments are "diverse" enough", skills bundling matters so that "separate productive attributes" command different prices in different sectors, they do not depart from the assumption that there exists a direct mapping from the characteristics of a person to the wage received. After rejecting empirically the hypothesis of uniform factor prices in US sectoral data, they propose factor immobility and non-linear hedonic pricing as alternative explanations, and even suggest that a linear characteristics pricing approach (Lancaster [24]) could still hold within sectors. The reason we wish to depart from the assumption of a direct mapping from characteristics to the wages observed, even accounting for bundling, is because it leads to an important puzzle in the analysis of skills gaps, mismatches, and shortages. If in equilibrium workers are paid according to the overall marginal product and scarcity of their bundles of skills within sectors, why don't rational people recognize the highly valuable types of sector-specific skills and acquire them, thereby eliminating shortage related arbitrage opportunities? Preferences, timing, costs are of course potential reasons. However, another possible explanation is that the hedonic approach does not fully account for the multifaceted role that skills diversity plays in the job matching process and the formation of competitive wages.

In the spatial competitive labour market model proposed below (Section 4), skills shortages can induce competition for those workers who possess the scarce bundles, but not necessarily so. The pricing is done through a completely different mechanism in which workers are not necessarily rewarded according to the value of their marginal products and the scarcity of their skills combinations.

To reach these conclusions, we start by recognising that, in reality, a firm could only hire a worker who has applied for its open vacancy. It is important to understand why workers apply to some jobs and not to others, and why different workers choose to apply to different numbers of vacancies. In particular, our goal is to investigate how skills heterogeneity influences such decisions.

Several approaches have been used in the economic literature to model the application process (first stage of the job matching). Undirected search models (Pissarides [29]) assume that workers and firms meet at random, thereby completely ignoring heterogeneity and the fact that workers should apply with a higher probability to jobs that would potentially give them higher utility. Directed search models (e.g. Moen [27], Galenianos & Kircher [15], Shimer [37]) do take this into account. However, they require workers to be able to observe all job offers, and design application strategies that are optimal given all other agents' strategies. Hence, when applying to jobs, workers must not only have perfect information, but also a certain level of strategic sophistication.

In order to avoid such unrealistic assumptions, we propose a novel approach that takes inspiration from the literature on spatial networks where link formation depends on distances (Janssen [20]). Specifically, we assume that for each worker, there exists a latent ranking of all vacancies which depends on skills mismatches and base wages that together determine ex-ante utility as defined in Section 4. The worker does not need to perfectly observe this latent ranking, whose sole function is to determine the probability distribution over the vacancies with which the worker applies to each of them. We also extend this basic set-up to allow for several and different numbers of applications per worker.

The simultaneous application decisions of all workers determine a bipartite network in which a link from a worker to an open vacancy corresponds to a job application. The competitive matching of workers and firms as well as the equilibrium wages depend on this network, as it determines the outside opportunities of both firms and workers. Firms that receive several applications for a given vacancy have to choose one among the different candidates, while job seekers who receive multiple offers can only accept one of them at the end of the negotiations.

Kranton & Minehart [21] provide one of the first analysis of competition on bipartite networks. However, the competitive mechanism devised in their paper would be inappropriate in our context because of heterogeneity. Although they show how competitive equilibrium outcomes are influenced by the whole structure of the bipartite buyer-seller network in which the outside opportunities "depend on the entire web of direct and indirect links"; the good exchanged in the process is homogeneous in the sense that a buyer's valuation for the good does not change depending on the seller from whom he acquires it. Similarly, sellers do not care about the buyers' identities, but only the price they receive for their good.

Instead, to model competition on the bipartite job applications network, we use a two-sided matching model. In existing economic theory, the matching literature which started with Gale & Shapley [14] probably provides the most general way of conceptualising heterogeneity in market-like settings. The competitive wage-adjustment mechanism we propose in Section 4 is related to the one in Crawford & Knoer [9]. Within the mechanism design literature, the main contribution of their paper is to recognize that in labour markets, agents' preferences need to be modelled as flexible because they can change over the negotiation process in which salaries adjust competitively. Despite being quite general, their model assumes perfect information, and as the authors argue themselves this is disadvantageous since "imperfect information is an essential characteristic of real labor market". Indeed, two-sided matching models usually involve algorithms that require each agent to be able to rank all agents on the opposite side, which is implausible in settings with large numbers of heterogeneous agents. This perfect information assumption might therefore be one of the main reasons why, despite their attractive and intuitive approach in such environments, two-sided matching models have not been more widely used in studying large real labour markets[2].

[2]They have been successfully applied in smaller settings, where agents on both sides can provide a complete ranking of all the agents on the opposite side of the market. See Roth and Sotomayor [32] for a textbook exposition of two-sided matching models and a discussion of their applications in the

Furthermore, even though heterogeneity in two-sided matching models has important implications for equilibrium outcomes, it remains unfounded, i.e., these models do not explain why some worker-firm pairs produce more or less, and yield more or less job satisfaction to the worker. They simply take pair-specific productivity and job satisfaction levels as exogenously given.

We try to address both issues by modelling skills diversity explicitly in Section 3, incorporating it directly into agents' preferences and job application decisions in Sections 4.1 and 4.2 respectively, then building a two-sided matching model on this spatial framework in Section 4.3.

Finally, our paper can also be related to the large empirical literature devoted to understanding skills mismatches and the related phenomena of under and over-education (cf., for instance, Sattinger [35] for a very detailed overview of the literature on qualitative mismatches, their causes and consequences), while Section 3.4 employs some ideas and techniques from the operations research literature (e.g. Eiselt & Sandblom [11]).

The discussion presented in this section shows that despite the "little attention [conventional] economic theory pays to the notion of skill" (UKCES [18]), often treating labour as a homogeneous good, many economists have actually pondered over skills and skills diversity/heterogeneity. The differences with our approach arise because the models reviewed above had been developed for different purposes.

Unfortunately, no clear and objective definitions of skills gaps, mismatches and shortages exist in the academic literature, where skills shortages, for instance, are often understood as a phenomenon that "causes vacancies to remain open longer" (Haskel & Martin [19]) and unfilled vacancies constitute "dynamic shortages", which only persist until wages have risen such as to make enough people acquire the scarce skills and bring the labour market into equilibrium once again (Arrow and Capron [4]). However, in practice, hiring difficulties, unfilled vacancies, wage rises, etc. are all potential *consequences* of shortages, not their proper *definition*. Hiring difficulties and unfilled vacancies may occur for reasons unrelated to shortages, like inefficient human resource recruiters, improper advertising of the job, etc., while raising wages is only *one of many* responses to shortages. For instance, the 2016/2017 Talent shortage survey conducted by ManpowerGroup [25] indicates that only 26% of employers respond to shortages by "paying higher salary packages to recruits". At the same time, 53% decide to "offer training and development to existing staff", 36% "recruit outside the talent pool", 28% "explore alternative sourcing strategies", 19% completely "change existing work models", etc.

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labour market for medical interns and college admissions mechanisms. Such models have also been used in "repugnant markets", e.g. kidney exchanges (Roth et al. [31]).

In the next section, we shall attempt to start filling this theoretical gap by proposing basic, geometric definitions of skills gaps, mismatches and shortages in a unified setting.

3. Modelling and Measuring Skills Diversity in Unity

The purpose of this section is to model skills diversity among workers and vacancies, and propose clear definitions of skills mismatches, gaps and shortages. Governments around the world have been concerned about skills gaps and mismatches and also want to minimize the numbers of unemployed workers and unfilled vacancies. In Section 3.4, we show that in an economy with a realistic degree of skills heterogeneity and no perfect coordination between the combinations of skills supplied and demanded, there will always be some positive minimum levels of skills gaps and mismatches if the objective is also to leave as few unmatched agents as possible.

3.1. **Participants and skills space.** The economy is composed of two finite and disjoint sets of open vacancies $\mathcal{V} = \{V_1, V_2, ..., V_M\}$, and job seekers (workers) $\mathcal{S} = \{S_1, S_2, ..., S_N\}$, with cardinalities $|\mathcal{V}| = M$ and $|\mathcal{S}| = N$ respectively. We use i and S_i, j and V_j interchangeably when referring to workers and vacancies respectively.

Consider an n-dimensional discrete skills space Ω, where each element is an $n \times 1$ skills vector $\vec{\omega} =< \omega_1, \omega_2, ..., \omega_n >$. Each component of the skills vector $\omega_l \in [0, \bar{\omega}_l]$, for $l = 1, ..., n$, corresponds to some specific type of skills (e.g. presentation skills, computer skills, teamwork, etc.). We shall assume that ω_l is discrete and varies between 0 (no l-type skills) and $\bar{\omega}_l$ (expert in l-type skills). For some types of skills, ω_l will be a binary variable $\{0, 1\}$ indicating whether or not the type of skills is possessed/required, while in other cases the value of ω_l will summarize the level of proficiency in the type of skills considered. Concisely, $\Omega = C^n \bigcap \mathbb{Z}^n$ where $C^n = [0, \bar{\omega}_1] \times [0, \bar{\omega}_2] \times ... \times [0, \bar{\omega}_n]$ for some positive integers $\bar{\omega}_l$, $l = 1, ..., n$.

We map both workers and vacancies onto this skills space, i.e. each worker $i \in \mathcal{S}$ corresponds to an n-dimensional non-negative vector of skills or capabilities $\vec{s_i} =< s_{1i}, s_{2i}, ..., s_{ni} >$ in Ω. Similarly, each vacancy $j \in \mathcal{V}$ corresponds to an n-dimensional vector $\vec{v}_j \in \Omega$ of skills required to perform job j, i.e. $\vec{v}_j$ is the skills vector that the benchmark candidate for vacancy j would possess. For simplicity, we assume that one firm is responsible for one vacancy only and therefore use the terms vacancy, job, firm, and employer interchangeably.[3]

[3]A possible extension for future research would be to consider more complex scenarios in which one firm simultaneously opens several vacancies, and can hire workers such as to compensate to some extent the skills deficiencies of ones by skills surpluses of others while still minimizing the overall skills gap of the whole team as defined in eq.3.2.

Note that although we use $\vec{s_i}$ and $\vec{v_j}$ when referring to workers and vacancies respectively, both types of vectors belong to the same skills space Ω. Moreover, the characterization of workers and vacancies could be made as precise as needed by simply increasing the dimensionality of the skills space; e.g. instead of just having "computer skills", we could include more skills types to capture proficiency with different types of computer software. In particular, uniqueness could be reached by setting a degree of heterogeneity $n < \infty$ such that no two workers and no two vacancies are the same. Although this is not necessary, we shall assume such a degree of skills diversity in section 4 in order to simplify some of the proofs, leaving the more general case as an extension for future research.

3.2. **Skills mismatches and skills gaps.** Consider an arbitrary job seeker i and an arbitrary vacancy j. As long as the vectors characterizing them in the skills space Ω do not coincide, it is possible to compute a distance between them. We define skills mismatches and skills gaps as two different distance measures:

Definition 1. The ***skills mismatch*** between worker $i \in \mathcal{S}$ and vacancy $j \in \mathcal{V}$, sm_{ij}, is the Euclidean distance on Ω between the vectors $\vec{s_i}$ and $\vec{v_j}$:

$$sm_{ij} = \| \vec{v_j} - \vec{s_i} \| = \sqrt{\sum_{l=1}^{n} (v_{lj} - s_{li})^2} \tag{3.1}$$

Since sm_{ij} is a measure of the *overall* distance between worker i and vacancy j, it increases both when the worker is overskilled and when he/she is underskilled in some skill type(s).

Definition 2. The ***skills gap*** between worker $i \in \mathcal{S}$ and vacancy $j \in \mathcal{V}$, sg_{ij}, is a measure of skills *deficiency*:

$$sg_{ij} = \sum_{l=1}^{n} \max\{0, v_{lj} - s_{li}\} \tag{3.2}$$

The skills gap only increases when the worker lacks some of the skills that are necessary for the job ($v_{lj} - s_{li} > 0$).

The pairwise skills mismatches and skills gaps between all workers and vacancies in the economy can be summarized by two NxM matrices **SM** and **SG**. For instance:

$$\mathbf{SG} = \begin{bmatrix} sg_{11} & \dots & sg_{1M} \\ \vdots & \ddots & \vdots \\ sg_{N1} & \dots & sg_{NM} \end{bmatrix}$$

i.e. the ith row of **SG** records the skills gaps of worker i with all open vacancies, whereas column j of **SG** contains the skills gaps that vacancy j has with respect to all job seekers.

Letting $\delta_l = v_{lj} - s_{li}$, it becomes clear that both sm_{ij} and sg_{ij} are specific cases of a more general distance measure defined as:

$$d_{ij} = \sum_{l=1}^{n} f(\delta_l) \tag{3.3}$$

where $f(.)$ is a monotonically increasing function in δ_l.

Skills gaps and skills mismatches therefore correspond to two different ways of perceiving and measuring divergences between the skills combinations embodied in the workers and those required by the open vacancies. Throughout the paper, we assume that workers care only about skills mismatches, since being employed in a job that matches their skills endowments more closely is both more satisfying and requires less extra effort. At the same time, employers are only concerned with skills gaps because any skills deficiency negatively affects their productivity.

Of course, in reality each agent probably has his/her own subjective perception of skills diversity, and the function $f(.)$ in eq.3.3 could be made agent and/or dimension-dependent to reflect this. For instance, when thinking about skills gaps, an employer might allow over-skills in some dimensions to compensate for under-skills in other ones, or penalize under-skilling in different dimensions differently. However, for simplicity and clarity purposes, we shall focus on skills mismatches and skills gaps as defined in equations 3.1 and 3.2.

A simple example that illustrates why it is important to separate skills mismatches from skills gaps is to consider a higher education graduate and two non-graduate vacancies: a barman and a plumber. Being a barman does not necessitate very specific skills, hence the skills gap between the higher education graduate and the barman vacancy is likely to be very small. However, the skills mismatch may be huge since the higher education graduate won't be able to use many of his skills if employed as a barman. By contrast, consider the higher education graduate and the plumber vacancy. This time, both the skills gap and the skills mismatch are likely to be large if the higher education graduate happens to know nothing about plumbing because a plumber is a non-graduate vacancy that requires specific skills.

3.3. **Measure space and skills shortages.** According to the British Government's Training Agency, a skills shortage occurs "when there are not enough people available with the skills needed to do the jobs which need to be done" (British Government's Training Agency [1]). Using the definitions introduced previously, a worker who has all the skills needed to do a particular job is someone who has a zero skills gap with this job. Eq.3.2 implies that such a *qualified* worker does not necessarily have to match a vacancy's requirements perfectly; he/she can be overskilled in some types of skills. Since a worker can therefore be qualified for many different jobs at the same time, the question of establishing whether or not there are "enough" qualified people available in the economy to "do the jobs which need to be done", i.e. to fill all open vacancies, seems rather non-trivial.

Indeed, contrary to skills mismatches and gaps, which are both measures that are specific to a certain worker-vacancy pair - i.e. i's skills gap and mismatch with vacancy j are unrelated to his/her skills gap and mismatch with a different vacancy h - the question of skills shortages cannot be treated in isolation. Hence, before proposing an objective condition that determines whether or not a vacancy experiences a skills shortage, we need to characterize the measure space of the economy in which vacancies and job seekers co-exist. This shall allow us to model their interdependence and conceptualize the "not enough" notion.

3.3.1. *Measure space.* The distribution of the combinations of skills available in the labour market (skills supply) defines a measure P on Ω. For instance, if the pool of job seekers is such that none of them has a specific combination of skills $\vec{\omega}$, i.e. $\vec{s}_i \neq \vec{\omega}$ for all $i \in \mathcal{S}$, this outcome will have measure zero under P, i.e. $P(\vec{\omega}) = 0$. Furthermore, this measure is such that:

$$P(\vec{\omega}) = |\{i \in \mathcal{S} | \vec{s}_i = \vec{\omega}\}| \tag{3.4}$$

where $|.|$ is the cardinality of the subset.

Hence the measure satisfies:

$$P(\Omega) = \sum_{\vec{\omega} \in \Omega} P(\vec{\omega}) = N$$

In a similar way, we can define a measure Q for the vectors of skills demanded to fill open vacancies. If none of the vacancies requires some combination of skills $\vec{\omega}$ - i.e. $\vec{v}_j \neq \vec{\omega}$ for all $j \in \mathcal{V}$, Q shall assign measure zero to this specific skills vector: $Q(\vec{\omega}) = 0$. Again, this measure is such that:

$$Q(\vec{\omega}) = |\{j \in \mathcal{V} | \vec{v}_j = \vec{\omega}\}| \tag{3.5}$$

and satisfies:

$$Q(\Omega) = \sum_{\vec{\omega} \in \Omega} Q(\vec{\omega}) = M$$

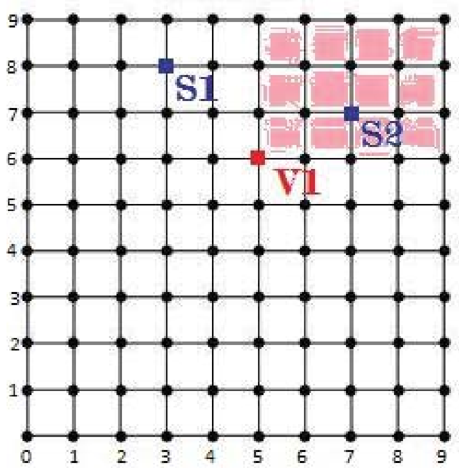
Consider the subset of all the job seekers whose skills gaps with vacancy j, as defined in eq.3.2, are zero. Their corresponding skills vectors lie in:

$$Z_j := \{\vec{\omega} \in \Omega | \omega_l \geqslant v_{lj}, \forall l = 1, ..., n\} \tag{3.6}$$

This includes the workers who possess exactly the $\vec{v}_j$ skills vector, as well as those who are overskilled in some type(s) of skills required by vacancy j but underskilled in none of them.

Let us call a job seeker i with $\vec{s}_i \in Z_j$ as *qualified* for vacancy j. Note that, even if the benchmark candidate for vacancy j is absent from the labour force ($P(\vec{v_j}) = 0$), vacancy j might still be able to hire a qualified worker as long as $P(Z_j) > 0$.

Figure 3.1 illustrates the idea in a two-dimensional space. The labour market is composed of one vacancy and two workers characterized by the vectors $\vec{v_1} =< 5, 6 >$, $\vec{s_1} =< 3, 8 >$, and $\vec{s_2} =< 7, 7 >$ respectively. The shaded area to the North-East of vacancy 1 corresponds to Z_1 as defined in eq.3.6. Only worker 2 ($\vec{s_2}$) belongs to Z_1, despite worker 1 ($\vec{s_1}$) being overskilled for vacancy 1 ($\vec{v_1}$) along the vertical dimension.



Notes: The shaded area marks the skills vectors of all the workers who would be qualified for vacancy V1. Only worker S2 has such a combination of skills. S1 is underskilled along the horizontal dimension.

Figure 3.1: Qualified workers

Let $\mathcal{Z}$ be the σ-algebra (collection of subsets of Ω) generated by the sets $Z_\omega := \{\vec{u} \in \Omega | u_l \geqslant \omega_l, \forall l = 1, ..., n\}$ for any $\vec{\omega} \in \Omega$.

The measure space for this economy is defined as the unique quadruple $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$.

As long as some workers are qualified for several vacancies at the same time, the question of whether or not a given vacancy is experiencing a skills shortage, in the sense of there not being enough qualified job seekers, cannot be addressed by looking at this specific vacancy in isolation. Instead, the vacancy has to be considered within the complete measure space characterizing the economy in which it operates $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$. Both the locations of all the other vacancies and the positions of all the job seekers matter when determining a skills shortage.

3.3.2. *Skills shortages.*

Definition 3. Vacancy j experiences a skills shortage in economy $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$ if:

$$Q(\vec{v}_j) + \sum_{\{\vec{\omega} \in \mathcal{H}_j\}} Q(\vec{\omega}) > P(Z_j) + \sum_{\{\vec{\omega} \in \mathcal{L}_j\}} P(\vec{\omega}) \tag{3.7}$$

where $\mathcal{H}_j := \{\vec{\omega} \in \Omega | P(Z_\omega \cap Z_j) \neq 0, \vec{\omega} \neq \vec{v}_j\}$ and $\mathcal{L}_j := \{\vec{\omega} \in \Omega | \vec{\omega} \in Z_u$ for $\vec{u} \in \mathcal{H}_j$, and $\vec{\omega} \notin Z_j\}$.

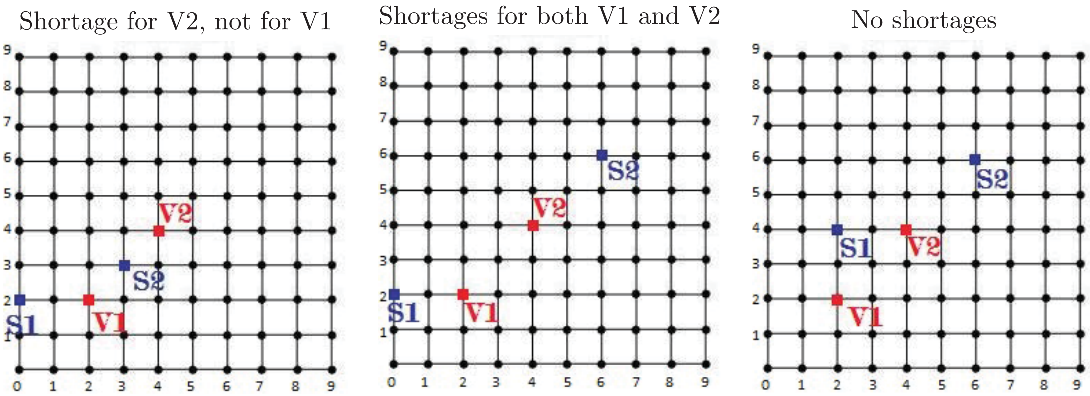
The left hand side of eq.3.7 gives the total demand for workers qualified for vacancy j. The first term is j's own demand. The second one sums up the demands from other firms in the same economy that also want to hire workers who are qualified for vacancy j (this is the subset of vacancies with skills vectors in $\mathcal{H}_j$). The first term on the right hand side is the total supply of workers qualified for vacancy j, while the second one adjusts this supply for the fact that firms with skills vectors in $\mathcal{H}_j$, i.e. which compete with j for the workers in Z_j, also have access to a pool of workers that are qualified for them but unqualified for vacancy j, and for which they do not compete with j.

To fully understand the condition for a skills shortage contained in eq.3.7, it is useful to look at an example with two dimensions where the problem can be inspected visually.

Figure 3.2 illustrates three possible scenarios with two vacancies and two workers on the same square lattice as the one introduced above.

Let the L, M, and R - subscripts stand for left, middle, and right panels of Figure 3.2. The economy depicted in the left wing panel can be summarized by the quadruple $\Psi^L = \{\Omega, \mathcal{Z}, P^L, Q^L\}$ where Ω is the square lattice, $P^L(\vec{s_1}) = P^L(\vec{s_2}) = 1$ with $\vec{s_1} =< 0, 2 >$, and $\vec{s_2} =< 3, 3 >$ and $P^L(\vec{\omega}) = 0$ for any other $\vec{\omega} \in \Omega$ such that $\vec{\omega} \neq \{\vec{s_1}, \vec{s_2}\}$. For the vacancies, $Q^L(\vec{v_1}) = Q^L(\vec{v_2}) = 1$ with $\vec{v_1} =< 2, 2 >$, and $\vec{v_2} =< 4, 4 >$ and $Q^L(\vec{\omega}) = 0$ for $\vec{\omega} \in \Omega$ such that $\vec{\omega} \neq \{\vec{v_1}, \vec{v_2}\}$. Furthermore, the qualified subsets are such that $P^L(Z_1) = 1$ and $P^L(Z_2) = 0$.

It is clear that vacancy 2 experiences a skills shortage since both potential applicants, S_1 and S_2, lack some of the skills required for V_2's benchmark combination $\vec{v_2}$. Since $P^L(Z_1 \cap Z_2) = 0$, eq.3.7 becomes: $1 + 0 > 0 + 0$ so that vacancy 2 experiences a shortage. By contrast, vacancy 1 does not experience a skills shortage when operating in Ψ^L. The qualified candidate is S_2, and there is *enough* of him/her because he/she



Notes: Three possible scenarios with two workers and two vacancies in a two-dimensional skills space are illustrated. As S2 becomes qualified for both vacancies (going from the left to the middle panel), there is no longer "enough" of him, so that both V1 and V2 experience skills shortages. Moving from the middle to the right panel, both shortages are eliminated by simply making S1 qualified for V1, so that there are enough qualified workers at the level of the economy to simultaneously fill both vacancies. Hence, the existence of shortages and the policies aimed at eliminating them cannot be considered in isolation.

Figure 3.2: Skills shortages

is not also qualified for vacancy 2. Eq.3.7 in this case gives $1 + 0 \leq 1 + 0$ since $Q^L(\vec{v_1}) = P^L(Z_1) = P^L(\vec{s_2}) = 1$ and the second terms on both sides are still equal to 0 because $P^L(Z_1 \cap Z_2) = 0$.

Suppose we change the location of S_2 from $< 3, 3 >$ to $< 6, 6 >$ while keeping everything else exactly the same as before. This is illustrated in the middle panel of Figure 3.2 . Worker 2 is now the *only* qualified candidate for both vacancies and so there is no longer *enough* of him/her. Indeed, now $P^M(Z_1 \cap Z_2) = 1$ and eq.3.7 becomes $1 + 1 > 1 + 0$ for vacancy 2, and also $1 + 1 > 1 + 0$ for vacancy 1, indicating a skills shortage for both of them.

Finally, the right wing of Figure 3.2 moves job seeker 1 from $\vec{s_1} =< 0, 2 >$ to $\vec{s_1} =< 2, 4 >$, keeping everything else as in the middle panel. S_1 is now in Z_1, i.e. qualified for vacancy 1, while still remaining outside Z_2. Graphically, it is obvious that there are no skills shortages at the level of the economy Ψ^R because there are *enough* qualified candidates to fill both vacancies simultaneously. Simply assign S_1 to V_1 and S_2 to V_2. The condition for a skills shortage in eq.3.7 is violated for both vacancies. For vacancy 2, the equation reads $1 + 1 \leq 1 + 1$ since $Q^R(\vec{v_2}) = 1$, $P^R(Z_2) = 1$, and $P^R(Z_1 \cap Z_2) = 1$. Note how important it is not to forget the right hand side adjustment $P^R(\vec{s_1}) = 1$. Indeed, although S_2 seems to be over-demanded since he/she is qualified for both vacancies so that total demand for him/her is $Q^R(\vec{v_1}) + Q^R(\vec{v_2}) = 2$, while his/her supply in the economy is only $P^R(\vec{s_2}) = 1$, it would be wrong to conclude that

V_2 experiences a skills shortage because S_2 is the only qualified applicant for it. The reason is that, contrary to the situation in Ψ^M, in economy Ψ^R vacancy 1 does have an alternative qualified candidate: S_1. This example illustrates why skills shortages can never be established in isolation because the "not enough" notion is defined relative to the space in which many heterogeneous vacancies and job seekers co-exist.

Another point to note when contrasting the middle and right panels of Figure 3.2 is that simply changing the location of worker 1 eliminates skills shortages for both vacancies. This has an interesting policy implication. In employers' surveys, financial services and engineering firms are often among those that are most concerned with skills shortages, but for different reasons (UKCES [18]). Financial companies often cite the lack of computer/problem solving skills and the lack of understanding of the finance industry as the two crucial deficiencies in their job applicants, while many engineering firms are concerned that their most gifted and best qualified candidates seek finance jobs because of the wage premium this industry is able to pay by leveraging talent (Célérier & Vallée [7]). Is having more graduates majoring in engineering the optimal solution in this case? Maybe not. Top engineers would continue flowing into finance, while lacking some important finance industry knowledge. A more appropriate solution might be to restructure finance/economics degrees so that students majoring in them, who potentially have more of the relevant finance background, also get more computer/problem solving skills. In terms of the spatial framework, this policy would correspond to changing the location of finance/economics graduates in the skills space in order to make them more attractive candidates for the financial industry. This would create more competition to the best engineering graduates, and force some of them back into seeking employment at engineering firms. The example is a caricature of reality, but a useful one to the extent that it illustrates how improving skills shortages for some industry might also alleviate skills shortages in another industry.

A corollary is that when choosing among policies directed at reducing skills shortages in different industries, an authority should include in its costs/benefits analysis the positive externalities that each policy might generate on other industries.

3.4. **Minimum levels of skills mismatches and skills gaps.** We now focus on determining the minimum levels of skills gaps and mismatches achievable in economy Ψ, when the objective is also to match as many workers and vacancies as possible. From a policy perspective, this is an interesting and important question since governments are usually not only concerned with reducing skills gaps and mismatches, but also want to leave as few unemployed workers and unfilled vacancies as possible.

We start with the following definitions:

Definition 4. An ***assignment*** or ***matching*** of workers to firms is a one-to-one correspondence $\mu : \mathcal{V} \bigcup \mathcal{S} \to \mathcal{V} \bigcup \mathcal{S}$ such that:

(1) $\mu(j) \in \mathcal{S} \bigcup \{j\}$ for any $j \in \mathcal{V}$;
(2) $\mu(i) \in \mathcal{V} \bigcup \{i\}$ for any $i \in \mathcal{S}$;
(3) $\mu(j) = i \Leftrightarrow \mu(i) = j$, $j \in \mathcal{V}$ and $i \in \mathcal{S}$, i.e. $\mu(\mu(j)) = j$.

The first two points ensure that a vacancy can either be assigned to a worker in $\mathcal{S}$ or left unfilled (assigned to itself: $\mu(j) = j$). Similarly, a worker can either be assigned to a vacancy in $\mathcal{V}$ or left unemployed (assigned to him/herself $\mu(i) = i$). The last point tells that if a vacancy is assigned to some worker, the worker has to be assigned to this specific vacancy. We will sometimes refer to $\mu(X)$ as the *match* of X for $X \in \mathcal{V} \bigcup \mathcal{S}$.

Moreover, every assignment μ has an associated assignment matrix $\mathbf{A} = [a_{ij}]$ with entries defined as:

$$a_{ij} = \begin{cases} 1 & \text{if } \mu(i) = j \\ 0 & \text{otherwise} \end{cases}$$

Definition 5. Two measures P and Q defined on a sample space Ω are ***equivalent*** if and only if whenever $P(\vec{\omega}) > 0$ we also have $Q(\vec{\omega}) > 0$ and vice versa for any $\vec{\omega} \in \Omega$.

Clearly, given a realistic degree of skills diversity in Ψ, and as long as the formation of skill combinations on the supply side is not perfectly coordinated with the sets of skills demanded, the two probability measures P and Q on Ω will not be equivalent.

If P and Q are not equivalent, at least some of the entries in the skills mismatch matrix $\mathbf{SM} = [sm_{ij}]$, where sm_{ij} is defined in eq.3.1, will be strictly positive. This implies that in an assignment which minimizes the number of unassigned agents, the minimum achievable sum of skills mismatches for the matched pairs - which we denote by SM^{min} - will not necessarily be zero.

To find SM^{min}, we solve a general assignment problem (Kuhn [23]) with an NxM cost matrix $\mathbf{SM}$. Since in general $N \neq M$ the problem is unbalanced. In labour markets, the number of unemployed usually outweighs the number of open vacancies, hence it is plausible to assume that $N > M$. The mathematical problem is then to "pick exactly one element in each [column] (fill each open vacancy) in such a way that each [row] (worker) is used at most once and that the total sum of the [M] elements thus chosen is minimal." (Eiselt & Sandblom [11]). To balance the problem, we introduce $(N - M)$ dummy open vacancies that have zero skills mismatches with all workers. This gives the transformed the NxN skills mismatch matrix $\widetilde{\mathbf{SM}}$. Any worker matched with a dummy vacancy in the final assignment will be considered as unmatched ($\mu(i) = i$).

The general assignment problem with NxN cost matrix $\widetilde{\mathbf{SM}}$ can be solved as the following linear programming problem:

$$\min \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{sm}_{ij} a_{ij} \tag{3.8}$$

subject to: (1) $\sum_{i=1}^{N} a_{ij} = 1$ for $j = 1, 2, ..., N$
(2) $\sum_{j=1}^{N} a_{ij} = 1$ for $i = 1, 2, ..., N$
(3) $a_{ij} = 1$ or 0 for all i, j

Several algorithms have been proposed to solve this problem. The Hungarian algorithm (Kuhn [23]) is the most famous and earliest one, but many other methods exist (cf. Dell'Amico & Toth [3] for an overview).

There can be several different assignments solving eq.3.8 subject to (1), (2), and (3), but every such assignment μ with assignment matrix $\mathbf{A} = [a_{ij}]$ is ***optimal*** in the sense that:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{sm}_{ij} a_{ij} \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{sm}_{ij} \hat{a}_{ij} \tag{3.9}$$

for any other assignment $\hat{\mu}$ with assignment matrix $\hat{\mathbf{A}} = [\hat{a}_{ij}]$ which also satisfies constrains (1), (2), and (3).

Since the dummy vacancies introduced have zero skills mismatches with all workers, the minimum skills mismatch achievable in the initial economy characterized by the measure space $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$ such that all M vacancies are filled can be computed as:

$$SM^{min} = \sum_{i=1}^{N} \sum_{j=1}^{N} \widetilde{sm}_{ij} a_{ij} \tag{3.10}$$

If all vacancies are to be filled, a lower level of overall skills mismatches could only be achieved by changing the locations of some agents. However, as soon as at least one worker or vacancy is moved, the measure space and hence the economy change. For a given $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$ and no unmatched vacancies, SM^{min} is therefore the minimum possible sum of skills mismatches for the matched pairs.

From a policy perspective, suppose all vacancies in a given economy Ψ are filled and the actual sum of skills mismatches is SM^R. Knowing SM^{min} will be useful since if $SM^R > SM^{min}$, the policy maker knows that he could achieve a lower overall skills mismatch level by simply reassigning existing workers among existing vacancies, i.e. the assignment itself must be inefficient in terms of skills mismatches. On the other hand, if $SM^R = SM^{min}$, the matching is already optimal since it also satisfies constrains (1), (2), and (3). A lower overall skills mismatch level could only be achieved by implementing policies that change the locations of workers and/or vacancies.

The same exercise could be performed with the skills gap matrix **SG** instead of **SM**. This would yield SG^{min} - the minimum skills gap level achievable in economy $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$ such that all vacancies are filled. From definitions 1 and 2, it is clear that $SG^{min} \leq SM^{min}$ since $sg_{ij} \leq sm_{ij}$ for all i, j. Moreover, if $\mathbf{SG} \neq \mathbf{SM}$, the optimal assignment(s) giving SM^{min} could be very different from those resulting in SG^{min}.

4. A Spatial Model of the Competitive Labour Market with Imperfect Information

The previous section created a framework for thinking about skills diversity and the way in which it is perceived by opposite sides of the labour market. We shall now investigate how skills gaps, mismatches, and shortages influence the job matching process and equilibrium outcomes in a competitive labour market with imperfect information. Imperfect information implies that neither workers nor firms are able to perfectly observe the measure space of the economy $\Psi = \{\Omega, \mathcal{Z}, P, Q\}$ in which they operate - an assumption that seems reasonable for a labour market with large numbers of open vacancies and job seekers at any point in time.

We start by determining how skills heterogeneity affects agents' preferences and payoffs, then model the job application and competitive wage adjustment processes, and discuss how skills heterogeneity influences competitive equilibrium outcomes.

4.1. **Payoffs and rankings.** As before, for simplicity, a firm operates exactly one vacancy and can only hire one worker. Similarly, a worker can only be employed in one vacancy.

Let w_{ij} be the wage that worker $i \in \mathcal{S}$ receives if employed in vacancy $j \in \mathcal{V}$. The determination of the competitive wage is discussed below. For this subsection it is enough to think of the wage as some positive real number.

As discussed in section 3, we continue to assume that, when considering skills heterogeneity, workers only care about skills mismatches, whereas firms are only affected by skills gaps.

Specifically, the profit of firm j that hires worker $i \in \mathcal{S}$ at wage w_{ij} is given by:

$$\pi_{ij} = p_j y_j(sg_{ij}) - w_{ij} \tag{4.1}$$

where $0 < p_j < \infty$ is the price charged by firm j and $0 \leq y_j(.) < \infty$ is firm j's specific production function which is monotonically decreasing in sg_{ij}. Moreover, let $\pi_{jj} = \psi_j$, i.e. when vacancy j remains unfilled, firm j receives an exogenously given finite amount ψ_j. This amount can either be interpreted as the cost (if $-\infty < \psi_j \leq 0$), or as the present value (if $\infty > \psi_j > 0$) of leaving vacancy j open. In either case, we take ψ_j as given, constant, and known by firm j at any point in time.

Definition 6. At a given wage w_{ij}, worker $i \in \mathcal{S}$ is ***acceptable*** to firm j if and only if $\pi_{ij} = p_j y_j(sg_{ij}) - w_{ij} > \psi_j$. Conversely, a worker is ***unacceptable*** if he/she is not acceptable.

The definition simply says that at a given wage a worker is acceptable to a firm if the firm would prefer to employ the worker at that wage rather than leave the vacancy open. Note that a worker can be acceptable at a given wage w, but become unacceptable at a higher wage $w' > w$.

The utility of worker $i \in \mathcal{S}$ employed in vacancy j at wage w_{ij} is summarized as:

$$u_{ij} = u_i(sm_{ij}, w_{ij}) \tag{4.2}$$

where $\frac{\partial u}{\partial w} > 0$ and $\frac{\partial u}{\partial sm} < 0$. Utility is decreasing in the skills mismatch sm_{ij} because being employed in a job that requires skills further away from his/her own combination of skills is both costlier in terms of effort and less satisfying for the worker. Let $u_{ii} = \kappa_i$ be the utility that a job seeker gets if he/she remains unemployed.

Definition 7. At a given wage w_{ij}, vacancy $j \in \mathcal{V}$ is ***acceptable*** to worker i if and only if $u_{ij} = u_i(sm_{ij}, w_{ij}) > \kappa_i$. Conversely, a vacancy is ***unacceptable*** if it is not acceptable.

Akin to definition 6, definition 7 just tells that a vacancy is acceptable to a worker at a given wage w_{ij} if and only if he/she prefers to be employed in that vacancy at w_{ij} instead of being unemployed and receiving utility $u_{ii} = \kappa_i$. Again, a vacancy can be acceptable to a worker at some wage w, but become unacceptable at a lower wage $w'' < w$.

We also assume that profits and utilities are independent across pairs, i.e. a firm does not directly care about the profits of another firm and a worker's utility is unrelated to the utilities of the other workers.

If worker i observes a given vector of wages $\vec{w_i} =< w_{i1}, w_{i2}, ..., w_{iM} >$ for all open vacancies, and the ith row of the skills mismatch matrix $\vec{sm_i}$, he/she can rank all the open vacancies $\mathcal{V}$ by utility. Let $\mathcal{R}(i, \vec{w_i})$ defined on the set $\mathcal{V} \bigcup \{i\}$ record this ranking.

For instance, suppose $M = 3$ and at the given vector of wages $\vec{w_i}$, worker i prefers vacancy V_2 to V_1 and would rather remain unemployed than work for V_3. The worker's ranking can be summarized as: $\mathcal{R}(i, \vec{w_i}) = \{V_2, V_1, S_i, V_3\}$.

Similarly, given a vector of wages $\vec{w_j} =< w_{1j}, w_{2j}, ..., w_{Nj} >$ and the jth column of the skills gap matrix $s\vec{g}_j$, firm j can rank all workers $\mathcal{S}$ in terms of profits. Let $\mathcal{R}(j, \vec{w_j})$ defined on the set $\mathcal{S} \bigcup \{j\}$ record this ranking.

We assume that the rankings $\mathcal{R}(i, \vec{w_i})$ for any $i \in \mathcal{S}$ and $\mathcal{R}(j, \vec{w_j})$ for any $j \in \mathcal{V}$ satisfy the properties of complete ordering and transitivity. For the workers, complete ordering implies that for any two open vacancies V_j and V_k, $j \neq k$, characterized by benchmark skill requirements $\vec{v}_j$ and $\vec{v}_k$ and offering wages w_{ij} and w_{ik} to worker i respectively, the worker can always rank the two vacancies, and say whether or not these vacancies are acceptable to him/her at the given wages. Transitivity implies that if worker i prefers V_j to V_k, and V_k to V_h, $j \neq k \neq h$, at the current wages offered, he/she must also prefer V_j to V_h. The explanations of complete ordering and transitivity are similar on the vacancies' side.

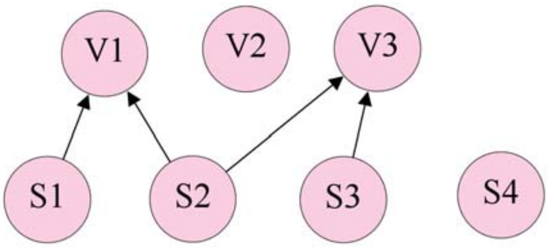
4.2. **Job application process.** The first step in the job matching process is the job application. For a firm to be able to hire a given worker, this worker must not only be seeking employment at the time when the firm opens its vacancy, but he/she must also apply to be considered for this open vacancy before the closing date. It is thus very important to understand what drives application decisions.

The outcomes of the application process can be summarized in a bipartite network. The nodes on the two opposite sides of this bipartite network correspond to the two finite disjoint sets of open vacancies $\mathcal{V} = \{V_1, V_2, ..., V_M\}$ and job seekers $\mathcal{S} = \{S_1, S_2, ..., S_N\}$. A directed link from $\mathcal{S}$ to $\mathcal{V}$ corresponds to a job application. The network therefore records the application decisions of all N workers and is the pictorial representation of the NxM incidence matrix/graph $\mathbf{B}$ with entries defined as:

$$b_{ij} = \begin{cases} 1 & \text{if } i \text{ applies to } j \\ 0 & \text{otherwise} \end{cases}$$

As an illustration, Fig. 4.1 shows a simple network with only three vacancies and four workers. The associated incidence matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



Notes: A link from a worker to a vacancy corresponds to a job application, e.g. S2 applies to both V1 and V3.

Figure 4.1: Bipartite job applications network

Modelling the application process is therefore equivalent to modelling the formation of the links in this bipartite network.

Links originate on the workers' side and are directed towards the set of open vacancies. Hence, those links, which lead to potential employment opportunities yielding higher utility levels for the workers, should be relatively more likely to occur. This paper assumes that workers' utility is increasing in wages and decreasing in skills mismatches. A worker should thus be more likely to apply to jobs that pay higher wages and/or with which he/she has a lower skills mismatch.

Specifically, suppose that there exists a base wage w_j for each firm j that can be publicly observed. Either the firm posts this wage together with the benchmark requirements $\vec{v}_j$ when opening its vacancy, or workers can recover it from representative datasets like the Annual Survey of Hours and Earnings (ONS), or websites like *Glassdoor* where people share anonymously the salaries that they are being paid in their jobs.[4] The base wage w_j therefore corresponds to the typical wage paid for this type of job and at this specific firm. Later, we shall also interpret w_j as the wage that a worker receives if employed in vacancy j in equilibrium such that there is no network-based competition for him/her. We take these base wages as given and exogenous, just as a job seeker would do when looking at what wages different firms typically pay.

We refer to the utility evaluated at the base wage as the *ex-ante* utility: $u_{ij} = u_i(sm_{ij}, w_j)$, to differentiate it from the *ex-post* utility which is evaluated at the final competitive wage w_{ij} that i receives in case he/she is matched with j in the competitive equilibrium.

With perfect information about Ψ and w_j, each worker i should be able to compute an *ex-ante* utility $u_{ij} = u_i(sm_{ij}, w_j)$ for all $j \in \mathcal{V}$ and therefore rank all open vacancies. However, in a large real labour market with imperfect information, it would be

[4]See http://www.glassdoor.co.uk/.

unreasonable to expect such a complete preference list from any job seeker. The worker might simply not be able to observe all suitable open vacancies. There could also be some uncertainty about either sm_{ij} or w_j.

To incorporate imperfect information, uncertainty, and unobserved intrinsic preferences into the application process, we propose a model for the formation of the bipartite job applications network that is inspired from the literature on spatial graphs. However, contrary to the standard spatial model in which "nodes are embedded in a metric space" and "link formation depends [only] on the relative position of nodes in the space" (Janssen [20]), we shall assume that link formation is influenced by overall ex-ante utility levels, and therefore not only the skills mismatches but also the base wages.

Intuitively, the probability of a link forming from S_i to V_j in the bipartite network, which corresponds to the probability with which worker i applies to vacancy j, shall be increasing in the utility that i would get if employed at j: $p_{ij} = p_i(u_i(sm_{ij}, w_j))$ with $\frac{\partial p}{\partial u} > 0$. We use the ex-ante utility because the worker cannot foresee the competitive wage he receives in equilibrium. To determine this wage, he/she would need to know how much competition there will be for him/her, if any. However, the latter will depend on the outside opportunities of the firms to which the worker applies - the other applicants for the same vacancies - which the worker cannot observe. Indeed, the worker is unaware of the application decisions of all the other job seekers because he/she does not know the complete measure space Ψ.

To some extent, the spatial labour market model presented here can be seen as a standard spatial model on the skills space, in which the relative positions of the nodes have been deformed by firms being able to pay different base wages. If wages did not enter the utility functions, workers would indeed be most likely to apply to those jobs that match their skills combinations the best. However, anecdotal evidence suggests that the base wage a person expects to be paid in a given job is an important determinant of the application decision. Workers are consciously willing to experience a higher degree of skills mismatch in exchange for a higher wage, and therefore base their application decisions on the overall ex-ante utility they could get in the job considered.

In order to overcome the requirement of perfect information, we assume that given w_j and $\vec{v}_j$ for all $j \in \mathcal{V}$, there does exist for each worker i, a latent ranking of vacancies $\mathcal{R}(i, \vec{w}_i)$. Nevertheless, this ranking does not have to be observed or known completely by the worker, since its sole function is to determine for each worker i, a latent application probability distribution over the set of vacancies $\mathcal{V}$. The only requirement is that application probabilities p_{ij} satisfy for all $i \in \mathcal{S}$ and $j \in \mathcal{V}$:

(1) $p_{ij} \in [0, 1]$;

(2) if $u_i(sm_{ik}, w_k) > u_i(sm_{ih}, w_h)$, $k \neq h$, then $p_{ik} > p_{ih}$;
(3) $p_{ik}(u_i(sm_{ik}, w_k)) = 0$ for any vacancy k that is unacceptable to worker i at w_k.

The resulting bipartite applications network is "self-organizing" because it is "formed by individual actions of autonomous agents" (Janssen [20]). Furthermore, despite being driven by skills mismatches and base wages through their effect on utilities and application probabilities, the job application process remains stochastic, thereby capturing unobserved individual intrinsic preferences for certain positions, uncertainty about either sm_{ij} or w_j, and/or information frictions (with positive probability workers are unaware of the best opportunities available to them in the labour market and hence do not apply for them).

Another characteristic that a realistic application process shall exhibit is a different number of applications per job seeker. Incorporating this feature and understanding why some workers only apply to one or two vacancies, while others apply to many, is important as this plays a crucial role in determining the outside opportunities of the firms and the workers in the competitive adjustment process analysed below. It is reasonable to assume that the cost of applying is decreasing in the skills mismatch (it is easier to write a cover letter for a job that requires your specific combination of skills). Workers are therefore more likely to apply to jobs where their skills mismatch is lower, which just reinforces the effect of skills mismatches on application probabilities that was already present through their effect on utility.

To allow for different numbers of applications per worker, we assume that, for each S_i, the set of links in the bipartite network (the application decisions) is formed by K random draws with replacement from the worker-specific application probabilities distribution over the set of vacancies $\mathcal{V} = \{V_1, V_2, ..., V_M\}$. K is some positive integer and we erase all multiple applications from one worker to the same vacancy. This last step not only yields the desired result of having different numbers of applications per worker, but also the following intuitive insight: those workers for whom there exists a small number of vacancies that provide much higher utility than all the other ones, will on average apply to fewer jobs. In some sense, for these workers there exists one best job profile that acts as a focal point and a larger part of their K applications will be allocated to applying to the job(s) that correspond(s) to this profile. Since any multiple applications are erased, the total number of their applications will be smaller than the average. On the contrary, for those workers who have a relatively general combination of skills and for whom there does not exist one best job profile, application decisions will be more widespread and numerous.

The importance of properly modelling and understanding the application process cannot be overstated. Indeed, competition for workers does not happen at the level of the economy, but exclusively on the bipartite network formed through the simultaneous application decisions of all workers. Hence, the bipartite applications network determines the outside opportunities of both the workers and the firms in the competitive adjustment process that leads to the local (network-based) equilibrium outcomes.

4.3. **Competitive wage adjustment.** In the bipartite applications network, whose formation was studied in the previous subsection, a firm potentially receives zero, one or several applications for its open vacancy. Similarly, a worker applies to zero, one or several open vacancies. The network therefore defines the outside opportunities for all agents. The objective is now to model competition for workers on this bipartite network. We start with several definitions, then propose a competitive wage adjustment process meant to mimic the way in which firms compete for the workers who have applied for their open vacancies.

In this section, in order to simplify some of the proofs, we shall assume that the level of heterogeneity - the n-dimensionality of the skills space Ω - is such that given a wage vector $\vec{w}$, a worker is never indifferent between two separate vacancies, or a vacancy and being unmatched. Similarly, given $\vec{w}$, an employer can always determine whether or not a candidate is acceptable, and for any two acceptable candidates, say which one he prefers.[5]

Definition 8. An ***individually rational outcome*** of the labour market is a one-to-one assignment of workers to vacancies μ and a wage vector $\vec{w}$ such that:

- At $\vec{w}$, workers are acceptable to the vacancies they are assigned to at μ:

$$\pi_{\mu(j)j} = p_j y_j(sg_{\mu(j)j}) - w_{\mu(j)j} \geq \psi_j$$

for all $j \in \mathcal{V}$, and $\pi_{\mu(j)j} = \psi_j$ iff $\mu(j) = j$;

- At $\vec{w}$, vacancies are acceptable to the workers they are assigned to at μ:

$$u_{i\mu(i)} = u_i(sm_{i\mu(i)}, w_{i\mu(i)}) \geq \kappa_i$$

for all $i \in \mathcal{S}$, and $u_{i\mu(i)} = \kappa_i$ iff $\mu(i) = i$.

Definition 9. An outcome in the ***core*** is an individually rational outcome $(\mu, \vec{w})$ such that no worker-vacancy pair (i, j), with $\mu(i) \neq j$, can negotiate a salary $\widetilde{w}_{ij}$ such that:

- S_i prefers V_j at $\widetilde{w}_{ij}$ to his/her match at $(\mu, \vec{w})$:

[5] A more general treatment could be undertaken in future research, although this assumption is probably not too far away from reality; when comparing two alternatives, it is often possible to find one small extra characteristic that will make us decide in favour of one or the other.

$$u_i(sm_{ij}, \widetilde{w}_{ij}) > u_i(sm_{i\mu(i)}, w_{i\mu(i)});$$

- V_j prefers S_i at $\widetilde{w}_{ij}$ to its match at $(\mu, \vec{w})$:

$$\pi_{ij} = p_j y_j(sg_{ij}) - \widetilde{w}_{ij} > \pi_{\mu(j)j} = p_j y_j(sg_{\mu(j)j}) - w_{\mu(j)j}.$$

Definition 10. An assignment μ ***respects*** an incidence matrix $\mathbf{B}$ if for all $i \in \mathcal{S}$ and $j \in \mathcal{V}$: if $\mu(i) = j$ then $b_{ij} = 1$.

In other words, an assignment respects an incidence matrix whenever a firm can only hire a worker who has previously applied for its open vacancy, i.e. there exists a directed link from this worker to the open vacancy in the corresponding bipartite applications network.

Fix an incidence matrix $\mathbf{B}$, formed as described in the previous subsection. The competitive wage adjustment on the corresponding bipartite network is a discrete N-dimensional time process $[\vec{w}_{ij}(t)]_j$, one for each $j \in \mathcal{V}$, with $w_{ij}(t) \in \mathbb{N}$ for any $t = 0, 1, ..., T$, during which wages evolve as follows:[6]

- $t = 0$:

$$w_{ij}(0) = \begin{cases} w_j & \text{if } b_{ij} = 1 \\ \infty & \text{otherwise} \end{cases}$$

i.e. firm j starts by considering all received applications at the base wage w_j. We assume that firm j never makes wage offers below this w_j for some institutional or reputational reasons which we do not investigate here. Given the resulting initial vector of wages $\vec{w}_{ij}(0)$ and the jth column of the skills gap matrix $s\vec{g}_j$, firm j can rank all acceptable applicants, if any, in terms of time zero profits:

$$\pi_{ij}(0) = p_j y_j(sg_{ij}) - w_{ij}(0)$$

Let the firm make a job offer to its best candidate at time $t = 0$, $B_j(0)$, if any, defined as:

$$B_j(0) = \begin{cases} \max_{\{i|b_{ij}=1\}} \pi_{ij}(0) = p_j y_j(sg_{ij}) - w_{ij}(0) & \text{if } \pi_{ij}(0) > \psi_j \\ \emptyset & \text{otherwise} \end{cases}$$

Note that if the firm receives no acceptable applications for its open vacancy, i.e. $\pi_{ij}(0) < \psi_j$ for all i such that $b_{ij} = 1$, it makes no offers at all, $B_j(0) = \emptyset$, leaves the vacancy open and gets $\pi_{jj} = \psi_j > -\infty$.

[6]Note that the unit of the wage does not matter. It is only necessary that in each time period if a wage rises, the increase is the same constant discrete amount for all firms and workers concerned. Similarly, there is no obvious time interpretation for T. It should simply be conceived as the number of steps necessary for the competitive wage adjustment to converge to an outcome in the core.

- $t \in [1, T-1]$:

A worker receives zero, one or several offers. Since in the application process a worker never applies to an unacceptable job and firms cannot offer wages lower than the base wages used by the worker to determine whether or not the vacancy is acceptable to him/her, the worker never receives unacceptable offers. Hence, we condition his/her choice only on the subset of firms which made him/her an offer in the previous step. The worker will *tentatively* hold the best job offer at time t, $H_i(t)$, if any:

$$H_i(t) = \begin{cases} \max_{\{j|B_j(t-1)=i\}} u_{ij}(t) = u_i(sm_{ij}, w_{ij}(t-1)) & \text{if } \{j|B_j(t-1)=i\} \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases} \tag{4.3}$$

The worker *tentatively* rejects the rest of the offers (if any). Until the very last period T all rejections and acceptances of offers are *tentative* because a tentatively rejected firm can come back to the same worker with a higher wage. Indeed, after each round of firm offers and worker decisions, wages adjust as follows:

$$w_{ij}(t) = \begin{cases} w_{ij}(t-1) + 1 & \text{if } B_j(t-1) = i \text{ and } H_i(t) \neq j \\ w_{ij}(t-1) & \text{otherwise} \end{cases} \tag{4.4}$$

i.e. the wage of worker i at firm j rises by one unit at time t only if firm j made an offer to i at $t-1$ and i tentatively rejected this offer.

Given the new vector of wages $\vec{w}_{ij}(t)$, firm j re-optimizes. It recomputes all profits and makes a job offer to its best candidate at time t, $B_j(t)$, if any:

$$B_j(t) = \begin{cases} \max_{\{i|b_{ij}=1\}} \pi_{ij}(t) = p_j y_j(sg_{ij}) - w_{ij}(t) & \text{if } \pi_{ij}(t) > \psi_j \\ \emptyset & \text{otherwise} \end{cases} \tag{4.5}$$

Note that if firm j made an offer at $t-1$ that was tentatively accepted at t, the problem at t is exactly the same as the one faced at $t-1$. Thus, the firm makes the same offer to the same worker at t, i.e. $B_j(t) = B_j(t-1)$. This means that a tentatively accepted offer remains valid until the worker rejects it for another offer that gives him/her higher utility, if he/she ever receives such a better offer before T. If this happens, the problem for the previously tentatively accepted firm changes as the wage of its previous best match increases. The firm re-optimizes and either chooses to come back to the same worker with a higher wage offering if at the new higher wage this worker is still its best alternative, or opts for the new best alternative, which could be leaving the vacancy open in case all workers (including the one that was the best match before the wage rise) become unacceptable at the new wage vector.

- $t = T$:

The competitive wage adjustment process stops when no tentative rejections are issued so that all wages converge: $w_{ij}(T) = w_{ij}(T-1)$ for all $i \in \mathcal{S}$ and $j \in \mathcal{V}$.

The outcome of this competitive wage adjustment process is a matching μ and a wage vector $\vec{w}$, such that:

$$\mu(i) = \begin{cases} j & \text{iff } B_j(T-1) = i \text{ and } H_i(T) = j \\ i & \text{otherwise} \end{cases}$$

$$\mu(j) = \begin{cases} i & \text{iff } B_j(T-1) = i \text{ and } H_i(T) = j \\ j & \text{otherwise} \end{cases}$$

Wages are only defined for the matched pairs, i.e.:

$$w_{ij}(T) = \begin{cases} w_{ij}(T-1) & \text{if } \mu(i) = j \\ \emptyset & \text{otherwise} \end{cases}$$

The payoffs on both sides can be summarized as:

$$u_{ij}(T) = \begin{cases} u_i(sm_{ij}, w_{ij}(T)) & \text{if } \mu(i) = j \\ \kappa_i & \text{otherwise} \end{cases}$$

$$\pi_{ij}(T) = \begin{cases} p_j y_j(sg_{ij}) - w_{ij}(T) & \text{if } \mu(j) = i \\ \psi_j & \text{otherwise} \end{cases}$$

Definition 11. A ***local competitive equilibrium*** on a bipartite network with associated incidence matrix $\mathbf{B}$ is an outcome $(\mu, \vec{w})$, such that $(\mu, \vec{w})$ is in the core and μ respects $\mathbf{B}$.

Theorem 12. *Fix a bipartite network with associated incidence matrix* $\mathbf{B}$. *The competitive wage adjustment process* $[\vec{w}_{ij}(t)]_j$, $j \in \mathcal{V}$, $t = 0, 1, ..., T$, *converges to a local competitive equilibrium in* $T < \infty$ *steps.*

To prove Theorem, we need to show that the outcome resulting from the competitive wage adjustment process $(\mu, \vec{w})$ is individually rational, in the core, and the assignment μ respects the given incidence matrix $\mathbf{B}$. We establish the proof of the theorem through a series of lemmas.

Lemma 13. *A worker who becomes unacceptable to a firm at step $t < T$, will never become acceptable to this firm at a later step $t' > t$.*

Proof. If worker i becomes unacceptable to firm j at $t < T$, it must be because $\pi_{ij}(t) = p_j y_j(sg_{ij}) - w_{ij}(t) < \psi_j$. By eq.4.4, at each iteration step in the wage adjustment process, i's wage can either remain constant or rise: $w_{ij}(t') \geq w_{ij}(t)$ for any $t' > t$. This implies that $\pi_{ij}(t') \leq \pi_{ij}(t) < \psi_j$ for any $t' > t$ and completes the proof. □

Lemma 14. *The wage adjustment process results in an assignment μ that respects the initially given bipartite applications network with associated incidence matrix* **B**.

Proof. The wage adjustment process starts by setting $w_{ij}(0) = \infty$ for all workers i such that $b_{ij} = 0$. This implies that any worker who has not applied for vacancy V_j in the initially fixed bipartite network becomes unacceptable already at $t = 0$ since for any such i, $\pi_{ij}(0) = -\infty < \psi_j$. By eq.4.5, a firm never makes an offer to an unacceptable worker and by Lemma 13 a worker that becomes unacceptable to a firm at some $t < T$, never becomes acceptable to this firm at a later $t' > t$. This implies that, during the wage adjustment process, firms only make offers, if any, to workers who had previously applied for their open vacancies, i.e. for which $b_{ij} = 1$. Any resulting assignment of workers to vacancies produced by the wage adjustment process will therefore respect **B** by construction. □

Lemma 15. *For any $i \in \mathcal{S}$ and $j \in \mathcal{V}$ such that $b_{ij} = 1$ and worker i is acceptable to firm j at w_j, the competitive wage adjustment process $\{w_{ij}(t)\}$ is bounded above.*

Proof. The process starts by setting $w_{ij}(0) = w_j$ for all $i \in \mathcal{S}$ and $j \in \mathcal{V}$ such that $b_{ij} = 1$. If worker i is acceptable to firm j at w_j, the firm could make one or several different wage offers to i during the wage adjustment process. The maximum wage that firm j could ever offer worker i is $p_j y_j(sg_{ij}) - \psi_j < \infty$. At any higher wage w_{ij}, worker i becomes unacceptable to firm j and by eq.4.5 a firm never makes offers to unacceptable candidates. Furthermore, by eq.4.3, a worker only rejects a previously tentatively accepted offer if he/she receives an offer from a different firm which gives him/her higher utility. Hence, the maximum wage w^*_{ij} that firm j would ever have to offer to worker i such that he/she never rejects its offer for another firm's offer, is such that:

$$u_i(sm_{ij}, w^*_{ij}) = \max_{k \neq j} u_i(sm_{ik}, p_k y_k(sg_{ik}) - \psi_k) + \varepsilon \tag{4.6}$$

where $\varepsilon \to 0$ and k is such that $b_{ik} = 1$ and i is acceptable to k at w_k. Since $p_k y_k(sg_{ik}) - \psi_k$ is finite for all such k, w^*_{ij} is also finite. Hence, for any $i \in \mathcal{S}$ and $j \in \mathcal{V}$, such that

$b_{ij} = 1$ and worker i is acceptable to firm j at w_j, the competitive wage adjustment process $\{w_{ij}(t)\}$ is bounded above by

$$\sup w_{ij} = \min\{p_j y_j(sg_{ij}) - \psi_j, w^*_{ij}\} \tag{4.7}$$

where w^*_{ij} is defined by eq.4.6. □

Lemma 16. *The wage adjustment process converges after a final number of steps T.*

Proof. The wage adjustment process converges as soon as no tentative rejections are issued and wages stop rising for all $i \in \mathcal{S}$ and $j \in \mathcal{V}$.

The wage adjustment process is constant for any $i \in \mathcal{S}$ and $j \in \mathcal{V}$ such that $b_{ij} = 0$ ($w_{ij}(t) = \infty$ for all t), or such that $b_{ij} = 1$ and worker i is unacceptable to firm j at w_j ($w_{ij}(t) = w_j$ for all t). This occurs because firm j never makes an offer to such a worker.

Hence, we just need to show that $\{w_{ij}(t)\}$ converges for any $i \in \mathcal{S}$ and $j \in \mathcal{V}$ such that $b_{ij} = 1$ and worker i is acceptable to firm j at w_j. For such (i, j) pairs, eq.4.4 implies that $\{w_{ij}(t)\}$ is monotonically increasing. By Lemma 15, we also know that $\{w_{ij}(t)\}$ is bounded above by $\sup w_{ij}$ as defined in eq.4.7. Since there are finitely many links in the bipartite applications network on which competition for workers can happen, any $\{w_{ij}(t)\}$ will always either converge to its supremum $\sup w_{ij}$ after finitely many steps, or the wage will stop rising at a level below $\sup w_{ij}$ which depends on the amount of competition for worker i in the network. □

Lemma 17. *The outcome $(\mu, \vec{w})$ to which the competitive wage adjustment process converges is in the core.*

Proof. By Lemma 14, μ respects $\mathbf{B}$. It is trivial to show that μ is individually rational. Workers never apply for vacancies that are unacceptable to them at the base wages, and firms never offer wages below the base wages. Hence, μ always assigns workers to either themselves or to vacancies acceptable at $\vec{w}$. On the firms' side, eq.4.5 implies that firms never make offers to unacceptable applicants at any point during the wage adjustment process. This proves that $(\mu, \vec{w})$ is individually rational.

Suppose that $(\mu, \vec{w})$ is individually rational, but not in the core. For this, there must exist a worker-vacancy pair (i, j), with $\mu(i) \neq j$, that can negotiate a salary $\widetilde{w}_{ij}$ such that:

- S_i prefers V_j at $\widetilde{w}_{ij}$ to his/her match at $(\mu, \vec{w})$:

$$u_i(sm_{ij}, \widetilde{w}_{ij}) > u_i(sm_{i\mu(i)}, w_{i\mu(i)}) \tag{4.8}$$

- V_j prefers S_i at $\widetilde{w}_{ij}$ to its match at $(\mu, \vec{w})$:

$$\pi_{ij} = p_j y_j(sg_{ij}) - \widetilde{w}_{ij} > \pi_{\mu(j)j} = p_j y_j(sg_{\mu(j)j}) - w_{\mu(j)j} \tag{4.9}$$

Equation 4.9 implies that at some point in the adjustment process, firm j must have made an offer $\widetilde{w}_{ij}$ to i. By eq.4.5 a firm's decision problem remains the same over time unless the worker rejects its offer. Therefore i must have rejected j's offer at $\widetilde{w}_{ij}$ since otherwise, j would never have made an offer to $\mu(j)$. By eq.4.3, workers reject an offer only if they receive a competing offer that gives higher utility. This implies that i could only have rejected j's offer at $\widetilde{w}_{ij}$ because he/she had a better offer w_{ik} from some firm k at that time. Furthermore, the same argument implies that i's final offer from firm $\mu(i)$ at $w_{i\mu(i)}$ must be at least as good as w_{ik} from k (with equality iff $\mu(i) = k$) :

$$u_i(sm_{i\mu(i)}, w_{i\mu(i)}) \geq u_i(sm_{ik}, w_{ik}) > u_i(sm_{ij}, \widetilde{w}_{ij})$$

which contradicts eq.4.8 and proves Lemma 17.

Lemma 17 completes the proof of Theorem 12. $\square$

4.4. **Local competitive equilibrium outcomes.** The impact of skills heterogeneity on wages for those workers who are matched to vacancies in a local equilibrium can be investigated by looking at the difference between the competitive wage they receive and the base wage for their job: $w_{ij}(T) - w_j$.

First, consider the case when $w_{ij}(T) = w_j$. This could happen for several reasons: worker i applied to one or only a few jobs because his/her utility is very convex in skills mismatches. Any small skills mismatch reduces utility by a lot. For this worker there exists a specific job profile to which he/she applies most of the time, thereby willingly constraining any potential future network-based competition for his/her skills. As a real world example, think about a PhD mathematician who decides to apply only for academic positions, thereby constraining himself any potential competition that there could be for his skills in industry or quantitative finance jobs and that could have pushed his wage above base levels.

Another scenario for $w_{ij}(T) = w_j$ is someone who has a rather general background and applied to many different jobs. His skills gaps are relatively large with most of the vacancies to which he applied so that he is not a candidate for whom there would be a lot of competition (as his wage starts rising he soon becomes unacceptable to many firms to which he applied). This worker receives one offer from some firm that did not get any better candidatures and hires him at the base level.

Finally, $w_{ij}(T) = w_j$ could also happen if i and j are close to a perfect match for each other, so that once i gets an offer from j he never wants to reject it for any other offer, even when j is just offering him the base wage. Potentially j is prepared to compete for i but this never happens because i never rejects j's offer.

On the other hand, $w_{ij}(T) > w_j$ indicates that there was at least one round of network-based competition for the worker, i.e. the worker tentatively rejected the offer of his final employer for a better one at least once in the negotiation process. For this to happen, the worker must himself be of high calibre so that he gets several offers from different employers, but also the worker's final employer must be in a situation where he does not have better alternatives. This could be the case if the vacancy is experiencing a skills shortage as defined in section 3.3, so that the rest of the applicants have relatively high skills gaps.

An important insight is that although employers in this model do not care about the profits of other rival firms, they care about both the skills gaps with their own applicants and the skills gaps that the rival firms experience in their applicants' pools. Consider a firm that receives several applications with only one of them being qualified for the vacancy. Even if it is able to hire the qualified candidate in a local equilibrium, larger skills gaps in other candidates imply that the firm will compete for the qualified worker more fiercely since the next best alternatives are not so attractive. If rival firms are in a similar situation and compete for the same qualified worker, his wage could rise well above the base level, eliminating most of the profit for his final employer.

Similarly, it is not necessary for some firm j to be experiencing a skills shortage (as defined in eq.3.7) in order to be induced to participate in network-based competition for its acceptable applicants and potentially lose all of them. The latter could occur if the firms competing with j for its qualified workers are willing and able to raise their wages sufficiently high. Vacancy j could also remain unfilled in a local competitive equilibrium if the workers qualified for j in the economy are relatively unlikely to apply to j. For instance, despite having zero skills gaps, the qualified workers could have substantial skills mismatches with j. Even if they do have relatively low skills mismatches, they could also be poached by other firms that offer higher base wages. Finally, an unfilled vacancy could simply be an unlucky realisation of the random application process.

5. Conclusion & Future Research

This paper designs a unified abstract framework that allows us to conceptualise skills gaps, mismatches, and shortages geometrically. We then propose a job matching model meant to mimic the real labour market. In a first step, skills mismatches influence the job application decisions of the workers, who do not have to possess the levels of information and strategic sophistication often assumed in standard economic models reviewed in Section 2. Job application decisions result in a bipartite network on which competition, shaped by skills gaps, mismatches and shortages, takes place in a second step.

The skills space, job application and competitive wage adjustment processes can all be simulated as part of an agent-based model, which in future research could be employed to further investigate how skills gaps, mismatches and shortages affect equilibrium outcomes in the context of skills diversity in unity and imperfect information.

Another potential direction for future research is to recreate empirically the measure space for the labour market of higher education graduates, i.e. project real world descriptions of graduates and relevant job openings, together with quantity data, onto a skills space whose dimension will be determined by how detailed the descriptions are. For this, we would need to assemble a detailed list of skills that graduates acquire while studying at university, i.e. create the skills space and construct the skills vectors characterizing higher education graduates. This could be done by looking at specific programme descriptions for each degree and university or using online datasources like LinkedIn, where people often provide detailed information on the courses taken and their own skills. To generate the measure of the combinations of skills supplied (measure P in Section 3.3), we would then need to get data on the actual numbers of students graduating in a given year by university and degree. A similar exercise would have to be conducted on the labour demand side, e.g. by using online vacancies data. The measure Q could be constructed from the numbers of job postings that specifically offer graduate employment or do not require substantial amount of work experience.

Once this is done, many interesting questions could be addressed and some policies could be tested employing the techniques developed in Section 3. For instance, we could find out objectively (i.e. without asking the employers themselves) which vacancies are experiencing skills shortages, what the minimum achievable levels of skills gaps and mismatches are, etc. This approach would also allow us to experiment with nation-wide or university-level curricula reforms, in order to see how they would reduce skills gaps, mismatches, shortages, and improve higher education graduates' employability.

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